# MAGNETOHYDRODYNAMIC FLOW PATTERNS IN A PULSATING MAGNETIC FIELD INVESTIGATED BY THE METHOD OF CHARACTERISTICS

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In recent years, relatively large magnetohydrodynamic (MHD) facilities with finite magnetic Reynolds numbers ($R_m$) and a magnetic field deformed appreciably by conducting flows have been built. The investigation of MHD devices such that $R_m \approx 1$ encounters difficulties involving the transients in MHD flows in a pulsating magnetic field, and characterized by substantial MHD interaction. Development of the method of characteristics is a natural approach to solving those problems.

In the general case, the method of characteristics is inapplicable to analysis of nonstationary MHD flow patterns with finite $R_m$ numbers, since the initial system of equations is not a hyperbolic system. However, if we introduce some restrictions which are allowable in a number of revelant cases, we can construct models of quasi-one-dimensional nonstationary MHD flow patterns with hyperbolic quasilinear initial systems of first-order equations for which the Cauchy data can be successfully formulated on the boundary curve of spatial or characteristic type [1].

Two types of problems of similar type are solved below for conducting flows in a pulsating transverse magnetic field. The first type encompasses supersonic flow patterns of gas through channels, the second type refers to the flow of free jets of incompressible fluid. The MHD interaction parameter and the $R_m$ number are assumed large. When certain restrictions are imposed on the geometry of the models and when the electric field in the channels is a potential field, the problems reduce to a Goursat problem, which can be handled in each case by an electronic digital computer, using the method of finite differences along the characteristic intervals.

1. Flow of a conducting gas through a narrow rectangular channel. An ideal perfect gas exhibiting a finite conductivity $\sigma$ flows at a supersonic velocity $\mathbf{u}(u, 0, 0)$ through a transverse magnetic field $\mathbf{B}(0, B, 0)$ along a channel with insulating walls $y = \pm y_0/2 = \text{const}$ and with ideally sectionalized electrode walls $z = \pm z_0/2 = \text{const}$. A unit area of the electrode walls corresponds to a closed external circuit of resistance R and inductance L situated beyond the exit from the channel. Because of the Faraday effect, an electric current of density $\mathbf{j}(0, 0, j)$ flows through the channel. All of the current taps are oriented in the positive direction of x, while the magnetic circuit with high magnetic permeability extends to the walls $\pm y_0/2$ on the outside, and closes between the entrance plane ($x = 0$) and exit plane ($x = l$) of the channel. With that geometry of the current leads and magnetic circuit, each elemental current at point x' induces its own magnetic field only in the region $x > x'$ [2], so that perturbations of the magnetic flux density do not propagate upstream. Similar conditions are fulfilled, with a certain approximation, in the case of narrow channels with no steel present, if the current taps are lined up with the flow velocity, and if the current flowing through the narrow channels makes the major contribution to the induced magnetic field.

The terminal effects are assumed to be suppressed, for instance, on account of the installation of longitudinal insulating baffles at the channel entrance and channel exit. The variables R, L, $\sigma$ are the arbitrary smooth functions R(x, t), L(x), and $\sigma$(x, t), respectively. The external magnetic field is independent of x and varies with time as $B_e = B_m \sin \omega t$. The electrodes at the channel entrance are open, $R(0, t) \to \infty$, so that $j(0, t) \equiv 0$, and the channel entrance parameters remain unperturbed. It is assumed that the basic contribution to the electric field $\mathbf{E}(0, 0, -E)$ is made by the drop in voltage across R and L, while the ro-

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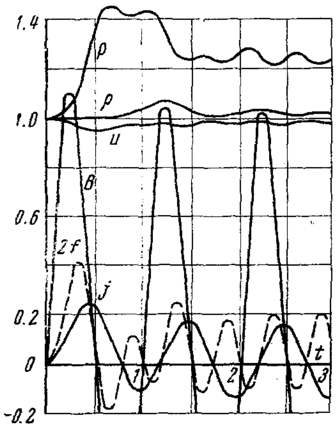


Fig. 1

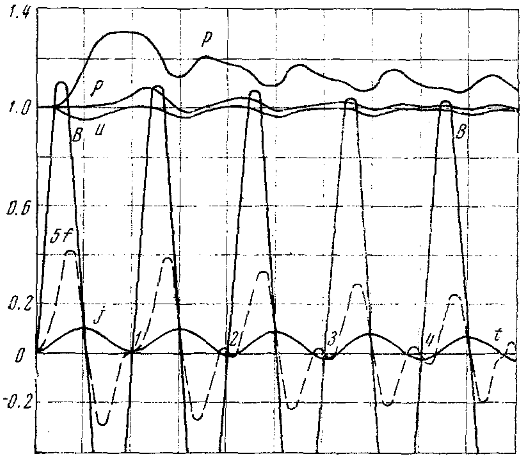


Fig. 2

tational (curl) electric field is small on account of $\partial B/\partial t$. The order of magnitude of the peak amplitude of the rotational component of **E** at the channel periphery is $\omega B_m z_0/2\,(1+z_0/l)$. If we take $uB_m$ as the characteristic amplitude of the potential electric field, then neglect of eddy currents in the channel will be justified at $\omega z_0/2u\,(1+z_0/l) \ll 1$ and at sufficiently large L.

The initial equations in the problem will be the continuity equations, the equations of motion, the energy equations, the equations of state, the first Maxwell equation, and the Kirchhoff law for a circuit of electrodes of unit area:

$$\frac{\partial\rho}{\partial t} - \frac{\partial}{\partial x}(\rho u) = 0, \qquad \rho\left(\frac{\partial u}{\partial t} + u\,\frac{\partial u}{\partial x}\right) + \frac{\partial p}{\partial x} = -jB$$

$$\rho\left(\frac{\partial e}{\partial t} + u\,\frac{\partial e}{\partial x}\right) + p\,\frac{\partial u}{\partial x} = \frac{j^2}{\sigma}$$

$$\frac{p}{\rho T} = \text{const}, \quad \frac{\partial B}{\partial x} = \mu_0 j, \quad L\,\frac{\partial j}{\partial t} + j\left(R + \frac{z_0}{\sigma}\right) = uB$$

Here e is the internal energy of the gas, and T is the temperature; the remaining notation is that commonly used.

We now proceed to the dimensionless variables, relating p, $\rho$, u, $\sigma$ to their values at the channel entrance $p_0$, $\rho_0$, $u_0$, $\sigma_0$, the flux density to $B_m$, the current density to $\sigma_0 u_0 B_m$, L to $\mu_0 z_0 l^2$, R to $z_0/\sigma_0$, x to $l$, and t to $l/u_0$.

We also introduce some similitude criteria: the magnetic Reynolds number $R_m = \mu_0\sigma_0 u_0 l$, the magnetohydrodynamic interaction parameter $S = \sigma_0 B_m^2 l/\rho_0 u_0$, the Mach number $M = u/\sqrt{\gamma p/\rho}$, where $\gamma$ is the adiabatic exponent, and the Euler number $E = p_0/\rho_0 u_0^2$. We then obtain a system of five equations in the unknowns u, p, $\rho$, j, B:

$$\frac{\partial\rho}{\partial t} + \rho\,\frac{\partial u}{\partial x} + u\,\frac{\partial\rho}{\partial x} = 0 \tag{1.1}$$

$$\rho\,\frac{\partial u}{\partial t} + \rho u\,\frac{\partial u}{\partial x} + E\,\frac{\partial p}{\partial x} = -SjB \tag{1.2}$$

$$\frac{\partial p}{\partial t} + u\,\frac{\partial p}{\partial x} - \frac{p}{\rho}\left(\frac{\partial\rho}{\partial t} + u\,\frac{\partial\rho}{\partial x}\right) + (\gamma-1)\,p\,\frac{\partial u}{\partial x} = \frac{S}{E}(\gamma-1)\,\frac{j^2}{\sigma} \tag{1.3}$$

$$\frac{\partial B}{\partial x} = R_m j \tag{1.4}$$

$$R_m L\,\frac{\partial j}{\partial t} + (R + \sigma^{-1})\,j = uB \tag{1.5}$$

We supplement this system with the equations for the total differentials of the unknowns,

$$du = \frac{\partial u}{\partial t}\,\partial t + \frac{\partial u}{\partial x}\,dx, \quad dp = \frac{\partial p}{\partial t}\,dt + \frac{\partial p}{\partial x}\,dx$$

and so on.

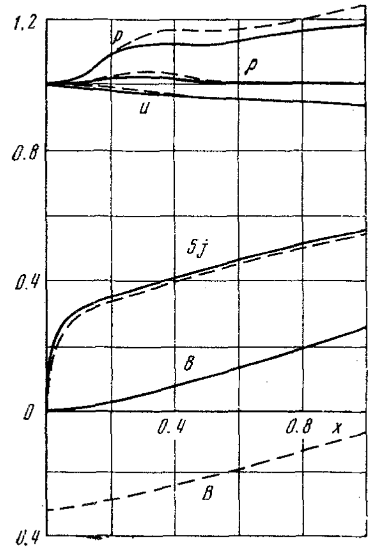


Fig. 3

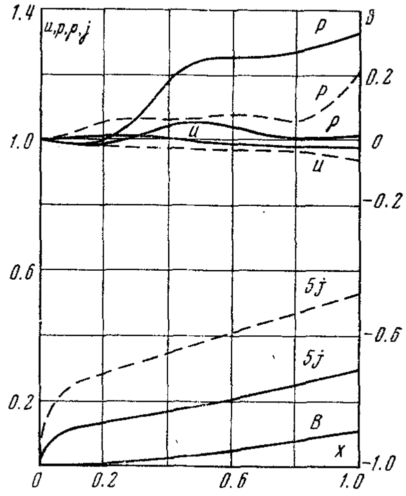


Fig. 4

Following the method presented by Shapiro [3], we now express each derivative $\partial u/\partial t$, $\partial u/\partial x$, $\partial p/\partial t$, $\partial p/\partial x$, ..., and so forth, in terms of the coefficients and free terms of the resulting system of ten equations according to Cramer's rule, and then set the numerator and denominator of each derivative equal to zero. Then the roots of the denominator will yield the characteristic directions dx/dt, and the roots of the numerator will yield the conditions for compatibility on the characteristics. In that way we arrive at the characteristic normal form [1] of the initial system, which is equivalent to the five ordinary differential equations acting along the characteristics:

$$E\,\frac{dp}{dt} + A\rho\,\frac{du}{dt} = Sj\left[(\gamma - 1)\,\frac{j}{\sigma} - AB\right] \quad \text{along} \quad \left(\frac{dx}{dt}\right)_{\mathrm{I}} = u + A \tag{1.6}$$

$$E\,\frac{dp}{dt} - A\rho\,\frac{du}{dt} = Sj\left[(\gamma - 1)\,\frac{j}{\sigma} + AB\right] \quad \text{along} \quad \left(\frac{dx}{dt}\right)_{\mathrm{II}} = u - A \tag{1.7}$$

$$E\,\frac{dp}{dt} - A^2\,\frac{d\rho}{dt} = (\gamma - 1)\,S\,\frac{j^2}{\sigma} \quad \text{along} \quad \left(\frac{dx}{dt}\right)_{\mathrm{III}} = u \tag{1.8}$$

$$\frac{dB}{dx} = R_m j \quad \text{along} \quad \left(\frac{dx}{dt}\right)_{\mathrm{IV}} = \infty \tag{1.9}$$

$$R_m L\,\frac{dj}{dt} = uB - j\,(R + \sigma^{-1}) \quad \text{along} \quad \left(\frac{dx}{dt}\right)_{\mathrm{V}} = 0 \qquad \left(A = \frac{u}{M}\right) \tag{1.10}$$

All of the characteristic directions dx/dt are real in the problem under discussion, so that the original system of equations (1.1)-(1.5) is hyperbolic. This conclusion is a natural one from a physical standpoint, since all of the perturbations in the model constructed propagate only downstream.

The characteristic system (1.6)-(1.10) can be obtained only with the aid of eigenvalues and the left eigenvectors of the matrix of the original system (1.1)-(1.5), if preceded by conversion to new variables $\eta = t + x$, $\tau = t - x$.

We shall assume that flow is steady-state in the channel when $t < 0$ and that $B_e = 0$, and at the instant $t = 0$, $B_e$ begins to vary in proportion to $\sin \omega t$. We can then formulate the initial conditions of the problem on the boundary curve in the xt plane, consisting of the positive semiaxes x and t. In effect, when $t = 0$, there is no magnetohydrodynamic interaction, and all of the parameters along the x axis are known from the previous steady state. Furthermore, when $x = 0$ we have $j \equiv 0$, and the channel entrance parameters remain unaltered, since none of the perturbations propagate upstream, by hypothesis.

Since all of the characteristics have a nonnegative slope dt/dx, the selected boundary curve allows us to construct a single-valued solution of the problem in the region of influence (the half-strip $0 \le x \le 1$, $0 \le t < \infty$) under the condition that the coefficients of the initial system be smoothly varying coefficients [1].

In view of the fact that the x axis and the t axis are characteristics, we thereby arrive at the Goursat problem. Its peculiar features are that the initial data cannot be specified arbitrarily on the boundary

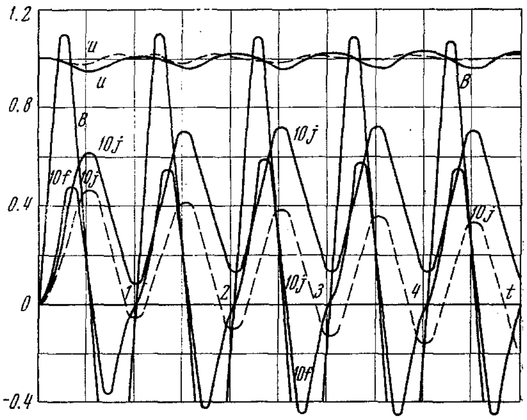

Fig. 5

curve, but must satisfy the appropriate characteristic equations, and in the case in point Eqs. (1.9) and (1.10), as well as the compatibility conditions at point (0, 0). It is readily seen that these restrictions are met. Equation (1.9) is satisfied identically, while Eq. (1.10) requires that

$$uB|_{x=0} = j(R + \sigma^{-1})|_{x=0} = \sin\omega t \quad \begin{matrix} R\to\infty \\ j\to 0 \end{matrix}$$

i.e., that the emf induced in open circuit be offset by a drop in the voltage from zero current across an infinite external resistance, as is physically evident.

It is to be noted that the restriction introduced, $R(0,t)\to\infty$, is not a stringent one, since the calculated origin of the x axis can be shifted somewhat upstream, when the continuous distribution R(x, t) is arbitrary, to where there are no electrons and no current, and R(x, t) can be approximated with an essential singularity, within the framework of the one-dimensional approximation.

The following conditions were assumed in the calculations:

$$u(0,t) = u(x,0) = p(0,t) = p(x,0) = \rho(0,t) = \rho(x,0) = 1$$
$$B(0,t) = \sin 2\pi t,\ B(x,0) = j(0,t) = j(x,0) = 0$$

The system (1.6)-(1.10) was solved on an electronic digital computer by the method of finite differences taken along the characteristic intervals [4]. The base grid formed by the characteristics IV and V, with mesh dimensions $\Delta x = \Delta t = 10^{-2}$ and $\Delta x = \Delta t = 10^{-3}$, was employed. The remaining characteristics were produced through the point with unknown parameters by linear interpolation of the data on the preceding computational step. Further fragmentation of the computational grid led only to a negligible further correction of the results, with the limits of 5%. The final results were verified in spot checks by direct substitution into the initial system (1.1)-(1.5) recorded in the form of finite differences on the intervals $\Delta x = \Delta t = 0.05$. The error amounted to $\approx 3\%$ of the terms in the equations greatest in absolute value, up to $t \cong 3$.

Figure 1 shows the distribution of parameters with respect to t for x = 0.89 at $\sigma = 1$, $L = R = x^{-1/5}$, $M_0 = 5$, $E = 0.024$, $R_m = S = 1$, and Fig. 2 shows the distribution under the same conditions, except for $R_m = S = 3$. In addition to the basic parameters, the curve of the electromagnetic force $f = jB$ is plotted here ($f > 0$ arbitrarily corresponds to a decelerating force, while $f < 0$ arbitrarily corresponds to an accelerating force).

Clearly, when the external sinusoidally varying field is switched on, there ensues at first an appreciable excursion of j(t) and B(t) into the upper half-plane, followed by a tendency to certain steady-state values. Typically, j(t) varies qualitatively in the same manner as in a conventional transient when a sinusoidal voltage is switched on across an inductive-resistive circuit (e.g., see [5]). When $\sigma_0$ increases, and concomitantly $R_m$ and S increase, the current increases slightly (the base $\sigma_0 u_0 B_m$ increases), but the averaged force $f$ in these instances declines because of the negative instantaneous values of $f(t)$, high in absolute value, and due to the more inductive character of the circuits. This may account for the lower primary burst of p(t) in response to large $R_m$ and large S. It is clear from Fig. 1 and from Fig. 2 that the increase in $R_m$ and in S also involves the transient, since the time constant of the circuits increases. The upward displacement of the B(t) curve with respect to the curve $B_e(t) = \sin 2\pi t$ is explained by the effect exerted by the induced magnetic field.

It is safe to assume that sufficiently short pressure surges in Fig. 1 and Fig. 2, in response to low L and appreciably large S, lead to discontinuous solutions not covered by the present analysis.

Fluctuations in the temperature of the gas are determined by the ratio $p/\rho$.

Curves of the distribution of parameters with respect to x are plotted in Fig. 3 and Fig. 4 for $R_m = S = 3$ and for different instants of time (Fig. 3 shows continuous curves plotted for t = 0.5 and dashed curves plotted for t = 0.55; Fig. 4 shows continuous curves plotted for t = 0.75 and dashed curves plotted for t = 1.5). The current density and the flux density increase with respect to x, while the p(x) curve and the $\rho(x)$ curve, together with the p(t) curve and $\rho(t)$ curve in Fig. 1 and Fig. 2, confirm the inferred formation of compression waves in the channel.

2. Motion of a free jet of incompressible inviscid conducting fluid. The jet advances with velocity $\mathbf{u}(u, 0, 0)$ and tangentially contacts the sides of the electrodes, sectionalized with respect to x, such that $z = \pm z_0/2 = \text{const}$, each pair of which is connected to an external circuit with ohmic resistance R' and inductance L'. In the region $y > y_0/2$, $y < -y_0/2$, we find a steel magnetic path ($\mu \to \infty$) instrumental in establishing the magnetic field $\mathbf{B}(0, B, 0)$. The geometry of the magnetic circuit and of the current taps is the same as in the preceding problem. The jet moves through the narrow clearance $y_0$ between the steel walls without touching them, i.e., $\delta < y_0$, where $\delta$ is the thickness of the jet with respect to y. A current $\mathbf{j}(0, 0, j)$ flows in the jet and establishes a field $\mathbf{E}(0, 0, -E)$ thanks to the voltage drop in the external circuits, a field which is much larger than the field due to electromagnetic induction in the channel. The end effects are eliminated by lengthwise baffles or by extending the magnetic field beyond the confines of the channel. The functions $R'(x, t)$ and $L'(x)$ are specified smooth functions at $x \geq 0$, $t \geq 0$; $R'(0, t) \to \infty$. The external magnetic field $B_e$ varies with time as $B_m \sin \omega t$. The gravitational and bulk dynamic forces act along the x axis, and their acceleration q(t) is specified. The conductivity of the jet $\sigma = \text{const}$.

The dynamics of the transient in the case of electromagnetic parameters is characterized in the one-dimensional approximation by the Kirchhoff equation for the circuit of one electrode pair

$$uBz_0 = I\left(\frac{z_0}{\sigma\Delta x\delta} + R'\right) + L'\frac{\partial I}{\partial t}$$

where $\Delta x$ is the longitudinal dimension of the electrode and $I = j\delta\Delta x$ is the current of the electrode pair.

When the continuity equation $u\delta \cong \text{const}$ is taken into account, we have

$$uBz_0 = j\left(\frac{z_0}{\sigma} + R'\Delta x\delta\right) + L'\Delta x\delta\left(\frac{\partial j}{\partial t} - \frac{j}{u}\frac{\partial u}{\partial t}\right) \tag{2.1}$$

The equation of motion of the jet is stated in the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{jB}{\rho} + q$$

The third initial equation will be the Maxwell equation

$$\partial B / \partial x = \mu_0 j \tag{2.2}$$

The notation for the last equation depends on the way B is averaged over the transverse cross section. The form of Eq. (2.2) corresponds to averaging of B over the cross section of the jet proper. But if B is averaged over the entire clearance $y_0$, then we have to write instead

$$\frac{\partial B}{\partial x} = \mu_0\frac{\delta}{y_0}j \tag{2.3}$$

In what follows, the form (2.2) will be used, even though the introduction of Eq. (2.3) instead of Eq. (2.2) would not complicate the problem in this method.

In what follows, we shall refer x to the channel length $l$, t to $l/u_0$, B to $B_m$, j to $\sigma u_0 B_m$, R' to $z_0/\sigma\Delta x\delta_0$, L' to $\mu_0 z_0 l^2/\Delta x\delta_0$, q to $u_0^2/l$, and we shall introduce the parameters

$$R_m = \mu_0\sigma u_0 l, \quad S = \frac{\sigma B_m{}^2 l}{\rho u_0}$$

where $\rho$ is the density of the fluid and $u_0$ and $\delta_0$ are the velocity and thickness of the jet at the channel entry.

Then the initial system of equations becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -SjB + q \tag{2.4}$$

$$\frac{\partial j}{\partial t} + j\frac{\partial u}{\partial x} = \frac{1}{L'R_m}[u^2B - j(u + R')] - \frac{Sj^2B}{u} + \frac{qj}{u} \tag{2.5}$$

$$\frac{\partial B}{\partial x} = R_m j \tag{2.6}$$

If we make use of Eq. (2.3) instead of Eq. (2.2), then the last equation appears in the form

$$\frac{\partial B}{\partial x} = R_m'\frac{j}{u} \qquad R_m' = R_m\frac{\delta_0}{y_0}$$

The system (2.4)-(2.6), as in the preceding problem, reduces to the normal characteristic form

$$\frac{du}{dt} = q - SjB \quad \text{along} \quad \left(\frac{dx}{dt}\right)_{\mathrm{I}} = u \tag{2.7}$$

$$\frac{dj}{dt} - \frac{j}{u}\frac{du}{dt} = \frac{1}{R_m L'}\left[u^2 B - j(u + R')\right] \quad \text{along} \quad \left(\frac{dx}{dt}\right)_{\mathrm{II}} = 0 \tag{2.8}$$

$$dB/dx = R_m j \quad \text{along} \quad (dx/dt)_{\mathrm{III}} = \infty \tag{2.9}$$

When $B_e \equiv 0$ at $t<0$, and $B_e = \sin \omega t$ at $t \geq 0$, then, as earlier, the initial conditions may be specified on the positive semiaxes x and t, which are characteristics II and III, and the problem reduces to the Goursat problem.

We assume here

$$u(0, t) = u(x, 0) = 1, \quad j(0, t) = j(x, 0) = 0$$
$$B(x, 0) = B_e(0) = 0, \quad B(0, t) = B_e = \sin 2\pi t \tag{2.10}$$

Conditions (2.10) are satisfied by corresponding additional restrictions in the characteristic problem, and are in agreement at the point (0,0).

In order to solve the normal form (2.7)-(2.9) with the initial conditions (2.10), we make use of the same method as in the first problem. The grid in the xt plane with cell dimensions $\Delta x = \Delta t = 10^{-2}$ (intervals along the II and III characteristics) was used. We assigned $q = 0$, $R_m = S = 5$, $R' = L' = x^{-1/5}$. The selective substitutions of the results for $t \leqslant 2$ in the initial system (2.4)-(2.6) yielded an error $\leqslant 3\%$ in the terms of highest absolute value.

Figure 5 shows how the parameters vary with respect to t when $x = 0.89$ (continuous curves) and when $x = 0.29$ (dashed curves), for $R_m = S = 5$. As in the first problem, the velocity lags slightly behind the oscillations of the electromagnetic force $f = jB$, and a more pronounced acceleration of the fluid by the force $f$ directed downstream is observed. The oscillations in u bring about an inverse change in $\delta$ and, accordingly, in the internal resistance between each pair of electrodes, and that has a telling effect on the transient process in the case of the electromagnetic parameters. That could be the explanation, specifically, for the behavior of the j(t) curve at $x = 0.89$, where the mean value rises initially to $t \approx 3$ and only later tends toward the t axis. In the initial portion of the jet ($x = 0.29$), the oscillations in u are much less pronounced, and the behavior of j(t) is the same as in familiar transient processes when variable voltage is switched on across inductances [5] (see also the dashed curves in Fig. 5).

Note that the method under discussion allows magnetic flux to be switched on in both problems with an arbitrary initial phase, i.e., $B_e = \sin(\omega t + \varphi_0)$. Actually, j (x, 0) cannot vary stepwise because of the inductances in the circuits, and the initial conditions on the $x > 0$ semiaxis are determined by the preceding steady state, under the condition that the role played by two-dimensional eddy currents in switching on the field $\varphi_0 \neq 0$ be negligible.

The analysis carried out also encompasses the case, for both flow patterns constantly in a pulsating magnetic field, when the electrodes were open at $t < 0$ and $j(x, t) \equiv 0$ throughout the channel, but where the external circuits with arbitrarily varying parameters were closed at $t \geq 0$.

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